University of Michigan
Space Physics Research Laboratory

Track Angle Calculation	CAGE No.	0TK63
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REVISION RECORD

Rev	Description	Date	Approval
C	• Describe the calculation of the orbit inclination • Revise the computation of tangent point position from Track angle	8 Nov 2002	
B	Changed arccot to arctan in formula of tangent point track angle in section 4	9 May 2002	
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APPROVAL RECORD

Function	Name	Signature	Date
Originator	D. Gell		
Analysis Software	M. Cooper		
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Instrument Scientist	W. Skinner		

Contents

Figures

List of Symbols

symbol	explanation
f	Earth flattening factor
H	spacecraft heading angle, the angle between the spacecraft velocity and North
$\vec{h}$	spacecraft specific orbital angular momentum
i	orbital inclination
$\hat{\ell}$	unit vector in the direction of the telescope line of sight
n	current orbit number
n_0	number of the first orbit of the day
o	offset of the tangent point from the orbit track
t	orbital track angle, sum of the true anomaly and the argument of perigee.
T	cumulative track angle measured from the first ascending node crossing of the day
v	true anomaly
x,y,z	components of the spacecraft position in the earth centered inertial reference frame
v_x, v_y, v_z	components of the spacecraft velocity in the earth centered inertial reference frame
□	longitude
□	right ascension of the ascending node
$□_G(t)$	Greenwich Sidereal Time at universal time t
$□_{s/c}$	right ascension of the spacecraft
□	telescope azimuth angle, measured in the spacecraft coordinate frame
□	co-latitude
□	telescope gimbal angle, measured positive downward from the spacecraft x-y plane
□′	geocentric latitude
□	geodetic latitude
□	angle between the velocity direction and East
p	orbital period
□	time
$□_□$	time of ascending node number i, where the first node crossing of the day is zero
□	argument of perigee

1. References

1) Gell, D.A., "Oblateness and Attitude Compensation", SPRL File 055-3662C, 21 April 1999

2. Introduction

The track angle is defined as the interior angle in the plane of the orbit from the ascending node to a point on the orbit. The geometry is shown in Figure 1. For tangent points, the track angle of a tangent point is defined as the track angle of the spacecraft when the point of interest is at an azimuth of ±90degrees. The method of calculating the track angle of the spacecraft and of tangent points is described below. The reverse calculations are described as well.

3. Spacecraft Track Angle Determination

3.1 Trigonometric Calculation

The angle between the ascending node and the position of the spacecraft can be determined by spherical trigonometric relationships as

$$t = \arcsin\left(\frac{\sin(\phi)}{\sin i}\right)$$

where the range of t is $-\pi/2 < t < \pi/2$. Since the track angle can run from 0 to 2π, the correct quadrant for t must be determined. This is accomplished by noting that the track angle is always in the same quadrant as the increment in longitude from the ascending node to the spacecraft. This increment in longitude is obtained from the right ascension of the spacecraft and of the ascending node as follows

$$\lambda_{s/c} = \arctan\left(\frac{y}{x}\right)$$

with $\lambda_{s/c}$ having a range of 0 to 2π. The longitude increment is then simply

$$\Delta\lambda = \lambda_{s/c} - \Omega$$

and t is adjusted to be in the same quadrant, using the following rules:

$$\begin{aligned}
&if\ 0 < \Delta\lambda \le \frac{\pi}{2} && do\ nothing \\
&if\ \frac{\pi}{2} < \Delta\lambda \le \frac{3\pi}{2} && replace\ t\ with\ \pi - t \\
&if\ \frac{3\pi}{2} < \Delta\lambda \le 2\pi && replace\ t\ with\ 2\pi + t
\end{aligned}$$

Finally, to determine the cumulative track angle, 2π per orbit is added to the orbital track angle:

$$T = t + 2\pi(n - n_0 + 1)$$

The addition of 1 to the term in parenthesis ensures that the value of the cumulative track angle is 2π at the first node crossing of the day.

If required the orbital inclination, i, may be determined from the spacecraft position and velocity. The orbital inclination is

$$\cos i = \frac{h_z}{\vec{h} \cdot \vec{h}}$$

where $\vec{h} = [h_x, h_y, h_z]^t$ is the orbital angular momentum,

$$\vec{h} = \vec{r}_{sc} \times \vec{v}_{sc}$$

Using the a trigonometric identity, and evaluating the cross product, the sine of the orbital inclination is

$$\sin i = \sqrt{1 - \cos^2 i}$$
$$= \sqrt{1 - \frac{\left(xv_y - yv_x\right)^2}{\left(yv_z - zv_y\right)^2 + \left(zv_x - xv_z\right)^2 + \left(xv_y - yv_x\right)^2}}$$

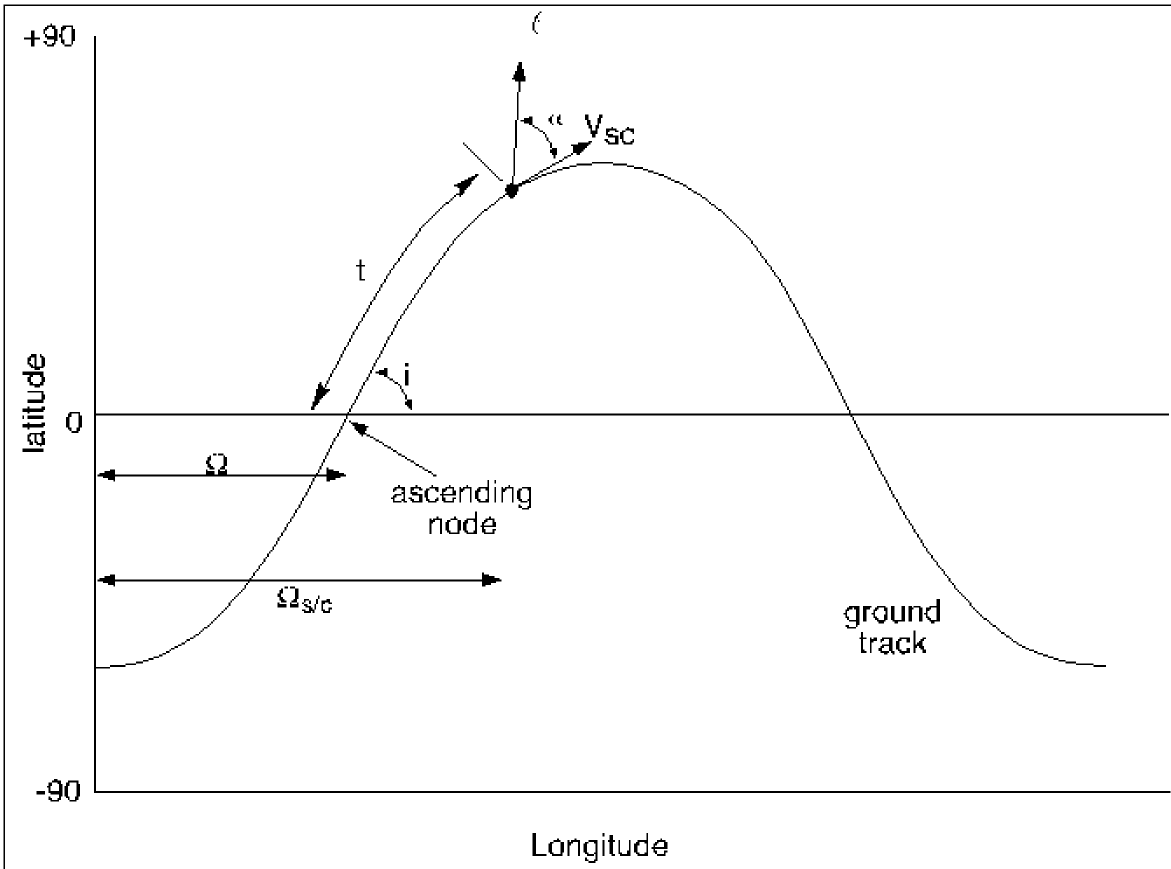

Figure 1, Track Angle Geometry

3.2 Time Based Calculation

For circular orbits, this calculation can be simplified by noting that the track angle increases uniformly with time. In this case, the cumulative track angle is

$$T = 2\pi\left(\frac{(\tau - \tau_i)}{p} + i + 1\right)$$

The addition of 1 to the term in parenthesis ensures that the value of the cumulative track angle is 2π at the first node crossing of the day.

This method has not been compared with the trigonometric method.

4. Tangent Point Track Angle

Since the tangent point is not on the orbit track, the track angle of some point on the orbit must be defined to be the tangent point track angle location. That point is chosen to be the one where the azimuth from the spacecraft to the tangent point is $\pi/2$. To calculate the tangent point track angle, the spacecraft track angle is first determined, then an increment is added. The increment can be determined using spherical trigonometry. The geometry is shown in Figure 2.

The angle between the line of sight and the orbit track is the measurement azimuth. The length of the side from the spacecraft to the tangent point is the elevation angle. From spherical trigonometry, the side Δt is

$$\Delta t = \arctan(\cos\alpha \bullet \tan\varepsilon)$$

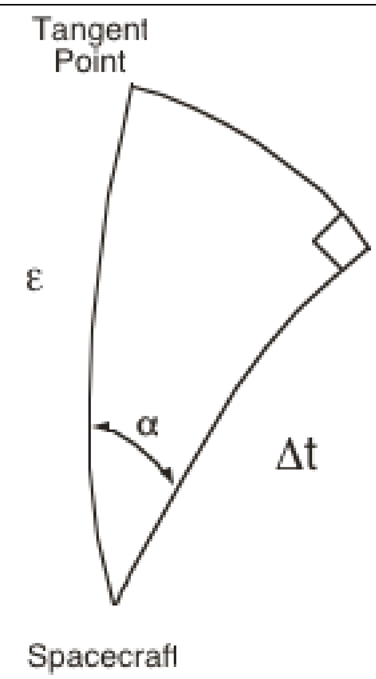

Figure 2, Tangent Point Track Angle

5. Tangent Point Location from Track Angle

The geographic location of the tangent point can be determined from the tangent point track angle, as defined above, the universal time, and the line of sight azimuth and elevation angles.

Referring to Figure 3, the problem is to determine the latitude and longitude of point M, the measurement tangent point, given the track angle of point X. In the figure, point S is the satellite location at the time a measurement at tangent point M was made. Point X is the track angle assigned to tangent point M. The points S,M and X form a right spherical triangle. The point P is the Earth's pole. θ_M and θ_X are the co-latitudes of points M and X respectively. The calculation of the tangent point position begins with calculating the position of point X from the track angle and universal time. Once that is obtained, the line of sight azimuth and elevation are used to determine o, the length of the arc between points X and M. The co-latitude of the tangent point is determined. Once that is known, the increment in longitude, $\Delta\lambda$, between the meridians of X and M is obtained and used, with the known longitude of X to determine the tangent point longitude.

The geocentric latitude of the track angle position X is

$$\phi_X = \arcsin(\sin i \sin T)$$

The longitude of the tangent point track angle is calculated by determining the right ascension of point X, as follows, then converting to earth centered coordinates. The right ascension of X is

$$\alpha_X = \Omega + \Delta\alpha$$

where Ω is the right ascension of the ascending node and $\Delta\alpha$ is

$$\Delta\alpha = \arctan\left(\frac{\cos i \sin T}{\cos T}\right)$$

The range of $\Delta\alpha$ is from 0 to 2π. Using the Greenwich Sidereal time at the time of interest t, the longitude of point X is

$$\lambda_X = \alpha_X - \theta_G(t)$$

The tangent point location, M, is obtained by first determining its co-latitude. The length of the arc MX is determined from the line of sight azimuth and elevation:

$$o = \arcsin(\sin\alpha \sin\varepsilon)$$

with representative azimuth of 45° and elevation of 22.9°, the arclength is 15.97°.

The angle between the meridian and the arc MX is obtained from the spacecraft heading angel at point X. The heading angle is a function of the track angle. In reference 1, the angle between the velocity vector and the east direction is given as:

$$\tan\psi = \pm\sqrt{\tan^2 i \cos^2 T\left(1 - \sin^2 i \sin^2 T\right)}$$

The positive sign is taken when the on the ascending leg of the orbit and the negative sign on the descending. Since he heading angle H as used here is $\pi/2 - \psi$ and the angle between the orbit track and the arc MX is a right angle, the required angle p is ψ.

$$p = \arctan\left(\pm\sqrt{\tan^2 i \cos^2 T\left(1 - \sin^2 i \sin^2 T\right)}\right)$$

The geocentric co-latitude of point M is obtained from the law of cosines for spherical triangles:

$$\cos\theta'_M = \cos\theta'_X \cos o + \sin\theta'_X \sin o \cos p$$

Noting the relationship between co-latitude and latitude, this equation may be re-written in terms of latitude:

$$\sin\phi'_M = \sin\phi'_X \cos o + \cos\phi'_X \sin o \cos p$$

The final step is to convert the geocentric latitude ϕ' to the geodetic latitude ϕ, as follows:

$$\tan\phi_M = \frac{\tan\phi'_M}{(1 - f)^2}$$

The longitude of the tangent point, λ_M, is simply

$$\lambda_M = \lambda_X + \Delta\lambda$$

where the longitude of point X is as shown above, and the longitude increment is

$$\Delta\lambda = \arcsin\left(\frac{\sin o \sin p}{\cos\phi_M}\right)$$

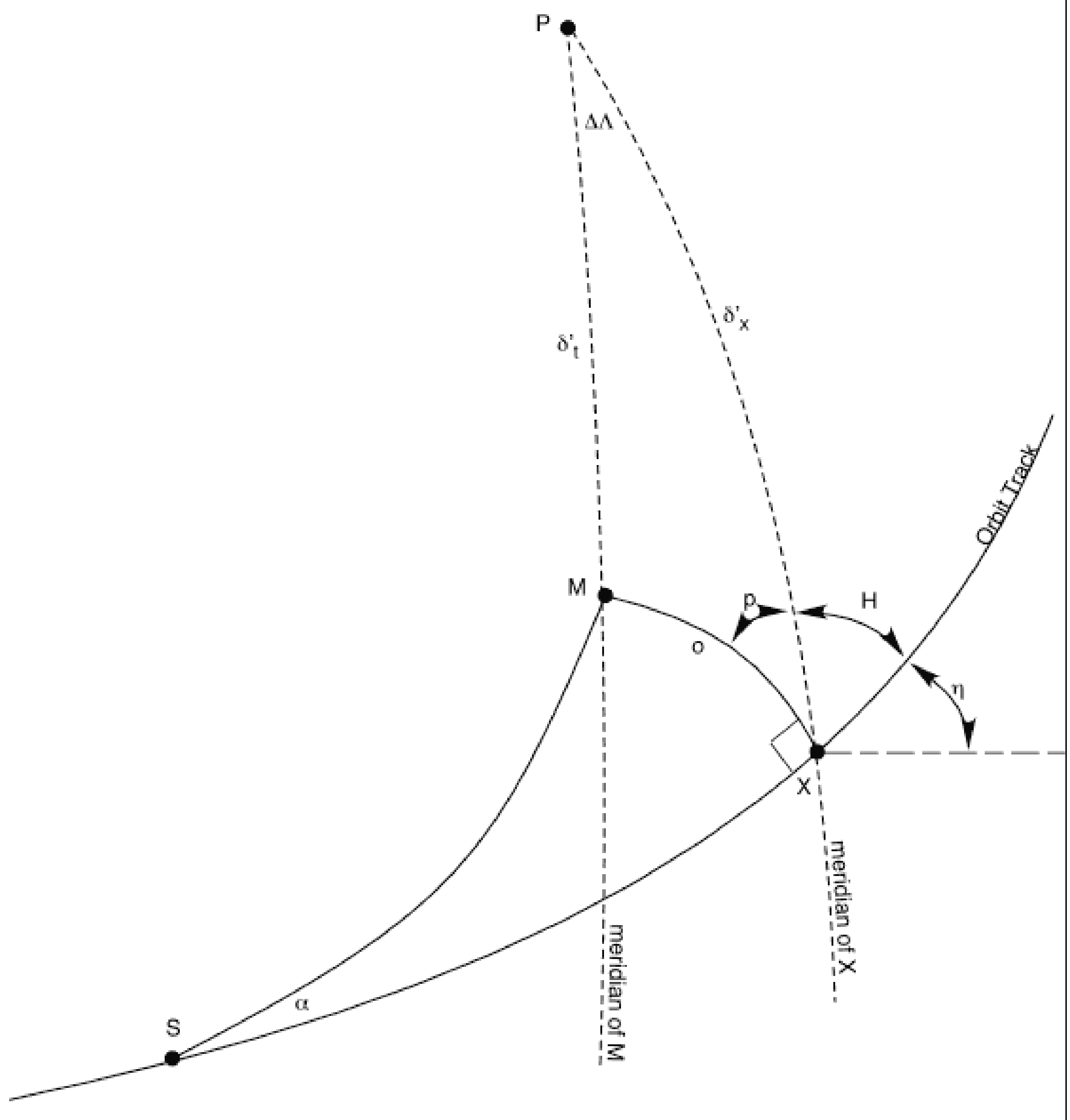

Figure 3, Tangent Point Latitude and Longitude